JOURNAL OF APPROXIMATION THEORY 53, 354–364 (1988)

On the Zeros of Lerch's Transcendental Function with Real Parameters

WOLFGANG GAWRONSKI

Abteilung für Mathematik, Universität Trier, D–5500 Trier, West Germany

AND

ULRICH STADTMÜLLER

Abteilung für Mathematik, Universität Ulm, D–7900 Ulm, West Germany

Communicated by G. Meinardus

Received February 20, 1986

1. INTRODUCTION AND SUMMARY

In this paper we deal with Lerch's transcendental function (cf. [6; 8, p. 33]) which can be defined by its power series

$$f_{\kappa,\lambda}(z) := \sum_{n=0}^{\infty} (n+\lambda)^{\kappa} z^n, \qquad \kappa, \lambda \in \mathbb{C}, \tag{1}$$

for $|z| < 1$; by analytic continuation it is seen to be holomorphic in the cut plane

$$\mathbb{C}^* := \{z \in \mathbb{C} \mid \text{if } \operatorname{Re} z \geqq 1, \text{ then } \operatorname{Im} z \neq 0\}. \tag{2}$$

Lerch's function plays an important role in various branches of pure and applied mathematics. For instance, it occurs in analytic number theory [6], summability [13, Chap. IV, 3], numerical analysis (e.g., [17]), and in the theory of structure of polymers [20]. In summability theory equivalence problems for Cesàro and certain discontinuous Riesz means require the number and the location of the zeros of $f_{\kappa,0}$ in $\mathbb{C}^*$ when κ is real. In approximation theory the convergence of cardinal Lagrange spline series with shifted interpolation grid $\lambda + \mathbb{Z}$, $\lambda \in [0, \frac{1}{2}]$, is closely related to some zeros of $f_{m,\lambda}$, where $m \in \mathbb{N}$ denotes the degree of the underlying

0021-9045/88 $3.00

Lagrange spline. In this case $f_{m,\lambda}$ is connected with the Euler–Frobenius polynomial, say $P_{m,\lambda}(z)$, by the relation

$$f_{m,\lambda}(z)=\left(\lambda+z\frac{d}{dz}\right)^m\frac{1}{1-z}=\frac{P_{m,\lambda}(z)}{(1-z)^{m+1}};\tag{3}$$

cf. [10; 11; 14, p. 7, Problem 46 if $\lambda=0$; 15–18]. In the sequel we suppose throughout that $\kappa>0$ and $\lambda\in[0,1)$.

It is known [4, 12] that all zeros of $f_{\kappa,\lambda}$ in $\mathbb{C}^*$ are real and $\leqq 0$. Moreover they are simple and $k+1$ in number if $k<\kappa\leqq k+1$, $k\in\mathbb{N}_0$. Other contributions to this particular question are contained in, e.g., [1–3, 5, 9–11, 18–21]. Hence we may assume the zeros $z_{\kappa,\nu}(\lambda)$ to be numbered according to

$$z_{\kappa,k}(\lambda)<z_{\kappa,k-1}(\lambda)<\cdots<z_{\kappa,1}(\lambda)<z_{\kappa,0}(\lambda)\leqq 0.\tag{4}$$

Among other asymptotic formulae in [3] we proved

$$z_{\kappa,\nu}(0)=-\exp\left(-\pi\cot\frac{2\nu+1}{\kappa+1}\frac{\pi}{2}\right)+\mathcal{O}(c^\kappa),\qquad \kappa\to\infty\tag{5}$$

except for "small" and "large" ν where $0<c<1$. Moreover this also holds for $\kappa\in\mathbb{C}$, $\kappa+1=(\kappa_0+1)(1+i\tau)$, $\kappa_0\to\infty$, $\tau\in\mathbb{R}$ fixed (compare also [18] for the special case $\kappa\in\mathbb{N}$, $\lambda=0$). In [16, 17] it was proved and indicated that the convergence of interpolating cardinal Lagrange spline series with grid $\lambda+\mathbb{Z}$, $\lambda\in[0,\frac{1}{2}]$, and degree $m\in\mathbb{N}$ is determined by its *radius of convergence*

$$R_m(\lambda):=\min(|\zeta_m(\lambda)|,|\zeta_m(1-\lambda)|),\tag{6}$$

where

$$\zeta_m(\lambda):=\max\{z_{m,\nu}(\lambda)\mid\nu=0,\ldots,m;\,z_{m,\nu}(\lambda)\leqq -1\}\tag{7}$$

denotes the so-called *main root* of $P_{m,\lambda}$. Using and modifying the results in [18], recently Reimer [17] obtained asymptotic formulae for $\zeta_{2r}(\frac{1}{2})$ and $\zeta_{2r+1}(0)$ as $r\to\infty$, giving in turn asymptotic estimates for $R_m(\lambda)$.

It is the main purpose of this paper to improve and to extend (5) by explicit inequalities for $z_{\kappa,\nu}(\lambda)$, that is, specifying the $\mathcal{O}$-term for $\kappa\geqq 1$ (Theorem 1). As an important consequence we get lower and upper estimates for the main root of the Euler–Frobenius polynomial and the radius of convergence of the cardinal Lagrange spline series, when $m\geqq 4$ (Theorem 3). As in [3] the proofs essentially are based on the so-called Lindelöf–Wirtinger expansion of $f_{\kappa,\lambda}$ giving a representation of the analytic extension onto $\mathbb{C}^*$ (Section 2). However, now the error estimates are made explicit.

2. Bounds for the Zeros

First modifying Lemma 1(iii) in [3] we derive the basic approximation formula for $f_{\kappa,\lambda}$ on the negative real axis. Applying residue calculus [6, 7, 22] or Poisson's sum formula (compare also [9; 8, p. 34]) to (1) we obtain the Lindelöf–Wirtinger expansion

$$f_{\kappa,\lambda}(z)=\frac{\Gamma(\kappa+1)}{z^{\lambda}}\sum_{m=-\infty}^{\infty}\frac{e^{2\pi im\lambda}}{(2\pi im+\log(1/z))^{\kappa+1}},\qquad \kappa>0 \tag{8}$$

giving the unique analytic extension onto $\mathbb{C}^*$. Here $\log 1/z$ is the principal branch in $\mathbb{C}^*$ meaning $\log 1/z$ is real for real positive z. Further, according to this choice we define $(u+iv)^{\kappa+1}=\exp((\kappa+1)\log(u+iv))$, where

$$\log(u+iv)=\tfrac{1}{2}\log(u^2+v^2)+i\arg(u+iv)$$

with

$$\arg(u+iv)=\begin{cases}\pi+\arctan(v/u), & u<0,\ v\geqq 0\\ \pi/2, & u=0,\ v>0\\ \arctan(v/u), & u>0\\ -\pi/2, & u=0,\ v<0\\ -\pi+\arctan(v/u) & u<0,\ v\leqq 0\end{cases}$$

and $-\pi/2<\arctan x<\pi/2$ for $x\in\mathbb{R}$ (see [3]). Putting $z=-r$, $r>0$ we write

$$\begin{aligned}f_{\kappa,\lambda}(-r)&=\frac{\Gamma(\kappa+1)}{r^{\lambda}}\sum_{m=-\infty}^{\infty}\frac{e^{(2m+1)\pi i\lambda}}{((2m+1)\pi i+\log(1/r))^{\kappa+1}}\\ &=\frac{\Gamma(\kappa+1)\,e^{\pi i\lambda}}{r^{\lambda}(\log(1/r)+i\pi)^{\kappa+1}}\{H_{\kappa,\lambda}(r)+R_{\kappa}(r)\}\end{aligned} \tag{9}$$

with

$$H_{\kappa,\lambda}(r):=1+\left(\frac{\log(1/r)+i\pi}{\log(1/r)-i\pi}\right)^{\kappa+1}e^{-2\pi i\lambda} \tag{10}$$

and

$$R_\kappa(r) := (\log(1/r) + i\pi)^{\kappa+1} \times \sum_{m=1}^{\infty} \left\{ \frac{e^{2\pi i m\lambda}}{(\log(1/r) + (2m+1)\pi i)^{\kappa+1}} + \frac{e^{-2\pi i(m+1)\lambda}}{(\log(1/r) - (2m+1)\pi i)^{\kappa+1}} \right\}. \tag{11}$$

Now the approximation of $f_{\kappa,\lambda}$ by $H_{\kappa,\lambda}$ is made precise by the following

LEMMA. *Suppose that $\delta > 0$, $\lambda \in [0,1)$, $\kappa \geqq 1$. Then, for $|\log r| \leqq 1/\delta$, we have*

$$f_{\kappa,\lambda}(-r) = \frac{\Gamma(\kappa+1)\, e^{\pi i\lambda}}{r^\lambda(\log(1/r) + i\pi)^{\kappa+1}} \{H_{\kappa,\lambda}(r) + R_\kappa(r)\}, \tag{12}$$

where

$$|R_\kappa(r)| \leqslant d(\delta, \kappa) := 4\left\{\left[\frac{1}{2\pi\delta}\sqrt{1+(3\pi\delta)^2}\right] + 1\right\} \times \left\{\frac{1+(\pi\delta)^2}{1+(3\pi\delta)^2}\right\}^{(\kappa+1)/2}. \tag{13}$$

For real x by $[x]$ we denote the largest integer not exceeding x.

Proof. We use the proof of Lemma 1(iii) in [3]. Since these estimations for the series in (11) are carried out term by term in absolute value, the independence of λ and in particular the inequality following formula (31)′ in [3, p. 281] can be used. Now (13) follows immediately.

Remark. For $\delta > 1/\sqrt{7}\,\pi$ we have

$$d(\delta, \kappa) \leqq 8\left(\frac{1+(\pi\delta)^2}{1+(3\pi\delta)^2}\right)^{(\kappa+1)/2} \leqq 8\left(\frac{1}{2}\right)^{(\kappa+1)/2}. \tag{13′}$$

Next, we compare the zeros $z_{\kappa,\nu}(\lambda)$ of $f_{\kappa,\lambda}$ (see (4)) with those of $H_{\kappa,\lambda}$ in

THEOREM 1. *Suppose that $\kappa \geqq 1$, $\delta > 0$, and $d(\delta, \kappa) \leqq 1$. Then*

$$z_{\kappa,\nu}(\lambda) = -\exp\left(-\pi \operatorname{cotan}\left(\frac{2(\nu+\lambda)+1}{\kappa+1}\frac{\pi}{2}\right)\right) + r_\kappa(\delta), \tag{14}$$

where

$$|r_\kappa(\delta)| \leqq \frac{2\pi^2}{\kappa+1} e^{1/\delta}\left(1+\frac{1}{(\pi\delta)^2}\right) \times\left\{\left[\frac{1}{2\pi\delta}\sqrt{1+(3\pi\delta)^2}\right]+1\right\}\times\left\{\frac{1+(\pi\delta)^2}{1+(3\pi\delta)^2}\right\}^{(\kappa+1)/2} \tag{15}$$

and $\nu \in I(\delta, \kappa)$ *with*

$$I(\delta,\kappa) := \left[-\lambda-\frac{1}{2}+\frac{1}{2}d(\delta,\kappa)+\frac{\kappa+1}{\pi}\operatorname{arccotan}\frac{1}{\pi\delta},\ (\kappa+1)\times\left(1-\frac{1}{\pi}\operatorname{arccotan}\frac{1}{\pi\delta}\right)-\lambda-\frac{1}{2}-\frac{1}{2}d(\delta,\kappa)\right]$$

$$(0 < \operatorname{arccotan} x < \pi \text{ for } x \in \mathbb{R}). \tag{16}$$

Remark.

$$J(\delta,\kappa) := \left[-\lambda+\frac{\kappa+1}{\pi}\operatorname{arccotan}\frac{1}{\pi\delta},\ (\kappa+1)\times\left(1-\frac{1}{\pi}\operatorname{arccotan}\frac{1}{\pi\delta}\right)-\lambda-1\right]\subset I(\delta,\kappa). \tag{16'}$$

Proof of Theorem 1. We use the lemma above and follow the proof of Theorem 1 in [3]. In view of (9) we put

$$z_{\kappa,\nu}(\lambda) = -\exp(-\pi \operatorname{cotan} x_{\kappa,\nu}(\lambda)),$$
$$x_{\kappa,\nu}(\lambda) := \frac{2(\nu+\lambda)+1}{\kappa+1}\frac{\pi}{2}-\varepsilon_{\kappa\nu}, \tag{17}$$

where $\varepsilon_{\kappa\nu} \in \mathbb{R}$ and $\nu \in \mathbb{N}$ have to be chosen suitably. The lemma requires

$$\pi|\operatorname{cotan} x_{\kappa,\nu}(\lambda)| \leqslant 1/\delta \qquad \text{and} \qquad x_{\kappa,\nu}(\lambda) \in (0,\pi) \tag{18}$$

and for these $\nu = \nu(\delta, \kappa)$ we obtain

$$f_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda)) = \frac{2\Gamma(\kappa+1)\sin^{\kappa+1} x_{\kappa,\nu}(\lambda)}{|z_{\kappa,\nu}(\lambda)|^\lambda \pi^{\kappa+1}} \times\left((-1)^\nu \sin((\kappa+1)\varepsilon_{\kappa\nu})+\frac{1}{2}R_\kappa(|z_{\kappa,\nu}(\lambda)|)\right).$$

Since $\sin(\pi d(\delta,\kappa)/2) \geqslant d(\delta,\kappa) > \frac{1}{2}R_\kappa\ (|z_{\kappa,\nu}(\lambda)|)$, $f_{\kappa,\lambda}$ changes sign exactly once in the interval

$$\left(-\exp\left(-\pi \operatorname{cotan}\left(\frac{\pi}{2(\kappa+1)}(2(v+\lambda)+1+d(\delta,\kappa))\right)\right),\right.$$
$$\left.-\exp\left(-\pi \operatorname{cotan}\left(\frac{\pi}{2(\kappa+1)}(2(v+\lambda)+1-d(\delta,\kappa))\right)\right)\right),$$

which gives $|\varepsilon_{\kappa v}| \leqslant (\pi/2(\kappa+1))\, d(\delta,\kappa)$. Now a straightforward computation yields that $v \in I(\delta,\kappa)$ satisfies (18). Finally, we apply the mean value theorem to obtain ($|\varepsilon'_{\kappa v}| \leqslant |\varepsilon_{\kappa v}|$)

$$|r_\kappa(\delta)| = |\varepsilon_{\kappa v}| \exp\left(-\pi \operatorname{cotan}\left(\frac{2(v+\lambda)+1}{\kappa+1}\frac{\pi}{2} - \varepsilon'_{\kappa v}\right)\right)$$
$$\times \frac{\pi}{\sin^2\left(\frac{2(v+\lambda)+1}{\kappa+1}\frac{\pi}{2} - \varepsilon'_{\kappa v}\right)}$$
$$\leqslant \frac{\pi^2}{2(\kappa+1)} e^{1/\delta} d(\delta,\kappa)\left(1+\frac{1}{(\pi\delta)^2}\right),$$

which implies (14) and (15).

Next, we turn to monotonicity properties of the zeros. The monotonicity of $z_{\kappa,v}(\lambda)$ with respect to λ was mentioned in [4, p. 220; 15] when $\kappa = m \in \mathbb{N}$. The monotonicity of $z_{\kappa,v}(0)$ with respect to κ was proved by Wirsing [21]. For the sake of completeness we treat the general cases in

THEOREM 2. *Assume the zeros $z_{\kappa,v}(\lambda)$ of $f_{\kappa,\lambda}$ to be numbered according to* (4).

(i) *If κ and v are fixed, then $z_{\kappa,v}(\lambda)$ is a strictly decreasing function of $\lambda \in [0,1)$.*

(ii) $z_{\kappa,0}(0) = 0$ *for all $\kappa > 0$. If $\lambda = 0$, $v \in \{1,\ldots,k\}$ or if $\lambda \in (0,1)$, $v \in \{0,\ldots,k\}$, then $z_{\kappa,v}(\lambda)$ is a strictly increasing function of $\kappa > 0$.*

Proof. This essentially is based on ideas in [21] for proving part (ii) if $\lambda = 0$. Therefore we restrict our considerations to some important steps.

In view of the implicit function theorem $z_{\kappa,v}(\lambda)$ possesses partial derivatives with respect to κ and λ. By (1), we have

$$z^{1-\lambda}\frac{\partial}{\partial z} z^\lambda f_{\kappa,\lambda}(z) = f_{\kappa+1,\lambda}(z), \qquad z \in \mathbb{C}^* \tag{19}$$

$$\frac{\partial}{\partial \lambda} f_{\kappa,\lambda}(z) = \kappa f_{\kappa-1,\lambda}(z), \qquad z \in \mathbb{C}^* \tag{20}$$

and

$$f_{\kappa,\lambda}(0)=\begin{cases}0, & \lambda=0,\\ \lambda, & \lambda>0,\end{cases}\qquad f'_{\kappa,\lambda}(0)=(1+\lambda)^{\kappa}>0. \tag{21}$$

From [4] we get $(k<\kappa\leqslant k+1)$

$$z_{\kappa,k}(\lambda)<\cdots<z_{\kappa,1}(\lambda)<z_{\kappa,0}(\lambda)\leqslant 0, \tag{4}$$

where $z_{\kappa,0}(\lambda)=0$ iff $\lambda=0$, by (21). Further, (9) and (1) imply

$$z^{\lambda}f_{\kappa,\lambda}(z)\to 0,\ \text{if } 0>z\to-\infty, \qquad z^{\lambda}f_{\kappa,\lambda}(z)\to 0,\ \text{if } z\to 0. \tag{22}$$

Next, Rolle's theorem combined with (19), (4), and (22) gives

$$\begin{aligned} z_{\kappa+1,\nu+1}(\lambda)<z_{\kappa,\nu}(\lambda)<z_{\kappa+1,\nu}(\lambda)<z_{\kappa,0}(\lambda)&\\ \leqslant z_{\kappa+1,0}(\lambda)\leqslant 0, \qquad \nu=1,...,k,& \end{aligned} \tag{23}$$

where both equalities hold iff $\lambda=0$. Since all $z_{\kappa,\nu}(\lambda)$ are simple, by (21), (19), (4), and (23), an immediate numbering yields

$$\begin{aligned} \operatorname{sign} f'_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda))&=(-1)^{\nu},\\ \operatorname{sign} f_{\kappa-1,\lambda}(z_{\kappa,\nu}(\lambda))&=(-1)^{\nu}, \qquad \nu=0,...,k. \end{aligned} \tag{24}$$

(i) From $f_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda))=0$ and (20) we get

$$\begin{aligned} 0&=\frac{d}{d\lambda}f_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda))\\ &=\kappa f_{\kappa-1,\lambda}(z_{\kappa,\nu}(\lambda))+f'_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda))\frac{\partial}{\partial\lambda}z_{\kappa,\nu}(\lambda), \end{aligned}$$

which in turn gives (use (24))

$$\operatorname{sign}\frac{\partial}{\partial\lambda}z_{\kappa,\nu}(\lambda)=-1, \qquad \nu=0,...,k.$$

(ii) Putting

$$f^{*}_{\kappa,\lambda}(z):=\frac{\partial}{\partial\kappa}f_{\kappa,\lambda}(z)$$

we get for $|z|<1$ that

$$\begin{aligned} f^{*}_{\kappa,\lambda}(z)&=\sum_{n=0}^{\infty}(n+\lambda)^{\kappa}\log(n+\lambda)\,z^{n}\\ &=\int_0^1 (f_{\kappa,\lambda}(z)-f_{\kappa,\lambda}(zt)\,t^{\lambda-1})\frac{dt}{\log(1/t)} \end{aligned} \tag{25}$$

also holding in $\mathbb{C}^*$ by analytic continuation. In order to show $\operatorname{sign}((\partial/\partial\kappa)\, z_{\kappa,\nu}(\lambda)) = 1$ in view of (24) and

$$0 = \frac{\partial}{\partial\kappa} f_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda)) = f^*_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda)) + f'_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda))\, \frac{\partial}{\partial\kappa} z_{\kappa,\nu}(\lambda)$$

it is sufficient to prove

$$\operatorname{sign} f^*_{\kappa,\lambda}(z_{\kappa,\nu}(\lambda)) = (-1)^{\nu+1}, \qquad \nu = 0, ..., k, \tag{26}$$

provided $z_{\kappa,\nu}(\lambda) < 0$. Writing $z_{\kappa,\nu} := z_{\kappa,\nu}(\lambda)$, by (25), we obtain

$$f^*_{\kappa,\lambda}(z_{\kappa,\nu}) = \frac{-1}{|z_{\kappa,\nu}|} \int_{z_{\kappa,\nu}}^{0} f_{\kappa,\lambda}(x)\, \frac{(x/z_{\kappa,\nu})^{\lambda-1}}{\log(x/z_{\kappa,\nu})}\, dx.$$

Proceeding in the very same manner as in [21], by splitting integral and by partial integration (use also (19)) successively we end with

$$f^*_{\kappa,\lambda}(z_{\kappa,\nu}) = \frac{-1}{|z_{\kappa,\nu}|} \sum_{j=0}^{\nu} (-1)^j j! \int_{z_{\kappa-j,\nu-j}}^{z_{\kappa-j-1,\nu-j-1}} f_{\kappa-j,\lambda}(x)$$
$$\times \frac{(x/z_{\kappa,\nu})^{\lambda-1}}{(\log(x/z_{\kappa,\nu}))^{j+1}}\, dx,$$

where $z_{\kappa-\nu-1,-1} := 0$. In view of (21), (23), and the simplicity of the zeros,

$$\operatorname{sign} f_{\kappa-j,\lambda}(x) = (-1)^{\nu-j},$$
$$\text{for } z_{\kappa-j,\nu-j} < x < z_{\kappa-j-1,\nu-j-1} < z_{\kappa-j,\nu-j-1}$$

and then (26) follows, which completes the proof.

3. Estimations of the Main Root

In this section we apply the preceding results to the main root of the Euler–Frobenius polynomials. Suppose throughout that $\kappa = m$ is a positive integer (see (3), (6), and (7)).

THEOREM 3. *With the above notations we have*

$$\text{(i)} \quad \zeta_m(\lambda) = z_{m,\nu(m)}(\lambda),$$

where

$$\nu(m) = \begin{cases} r, & m = 2r,\ \lambda \in [0, 1) \text{ or } m = 2r+1,\ \lambda \in [\tfrac{1}{2}, 1) \\ r+1, & m = 2r+1,\ \lambda \in [0, \tfrac{1}{2}) \end{cases} \tag{27}$$

and

$$z_{2r,r}(0) = -1, \qquad z_{2r+1,r}(1/2) = -1,$$

$$\text{(ii)} \quad \zeta_m(\lambda) = -\exp\left(-\pi \operatorname{cotan}\left(\frac{2(\nu(m)+\lambda)+1}{m+1}\frac{\pi}{2}\right)\right) + r_m,$$

where

$$|r_m| \leqslant \frac{4\pi^2}{(m+1)\sin^2\left(\frac{m-2}{m+1}\frac{\pi}{2}\right)} \exp\left(\pi \operatorname{cotan}\frac{m-2}{m+1}\frac{\pi}{2}\right)$$

$$\times\left(1+8\sin^2\left(\frac{m-2}{m+1}\frac{\pi}{2}\right)\right)^{-(m+1)/2}$$

for $m \geqslant 4$, *and*

$$|r_m| \leqslant \frac{16\pi^2 e^{\pi/\sqrt{3}}}{3(m+1)}\left(\frac{1}{7}\right)^{(m+1)/2}, \qquad \textit{for} \qquad m \geqslant 8,$$

$$\text{(iii)} \quad R_m(\lambda) = \begin{cases} |z_{2r,r}(\lambda)|, & m=2r,\ 0<\lambda\leqslant\frac{1}{2} \\ 1, & m=2r,\ \lambda=0 \text{ or } m=2r+1,\ \lambda=\frac{1}{2} \\ |z_{2r+1,r+1}(\lambda)|, & m=2r+1,\ 0\leqslant\lambda<\frac{1}{2}. \end{cases}$$

Proof. (i) From (9) we obtain

$$f_{\kappa,\lambda}(-1) = \frac{\Gamma(\kappa+1)}{\pi^{\kappa+1}}\, 2 \sum_{\nu=0}^{\infty} \frac{\cos\left((2\nu+1)\pi\lambda - \frac{\kappa+1}{2}\pi\right)}{(2\nu+1)^{\kappa+1}}$$

giving $P_{2r,0}(-1)=0$ and $P_{2r+1,1/2}(-1)=0$. Now using the relation

$$P_{m,\lambda}(z) = z^m P_{m,1-\lambda}(1/z)$$

(see, e.g., [5, 8, 11, 12, 20]) implying the zeros to be "reciprocal," a simple counting of $z_{m,\nu}(\lambda)$, and the monotonicity with respect to λ in Theorem 2(i), complete part (i).

(ii) We apply Theorem 1 above. In order to get good bounds for $\zeta_m(\lambda)$ we try to choose $\delta=\delta(m)$ as large as possible. In view of (16)′ we put

$$\delta(m) := \sup\{\delta>0 \mid \nu \in J(\delta, m)\}$$

and it follows that (observe (27))

$$\delta(m)=\begin{cases}\dfrac{1}{\pi}\tan\dfrac{r-\lambda}{2r+1}\pi, & m=2r,\ 0\leqslant\lambda<1\\[2ex] \dfrac{1}{\pi}\tan\dfrac{r+1-\lambda}{2r+2}\pi, & m=2r+1,\ \frac{1}{2}\leqslant\lambda<1\\[2ex] \dfrac{1}{\pi}\tan\dfrac{r-\lambda}{2r+2}\pi, & m=2r+1,\ 0\leqslant\lambda<\frac{1}{2}.\end{cases}$$

Choosing

$$\tilde{\delta}(m):=\frac{1}{\pi}\tan\frac{m-2}{m+1}\frac{\pi}{2}=\frac{1}{\pi}\tan x_m,$$

say, in all cases clearly we have $\tilde{\delta}(m)<\delta(m)$ and

$$\begin{aligned} d(\tilde{\delta}(m),m)&=4\left\{\left[\frac{1}{2}\left(\frac{1}{\sin^2 x_m}+8\right)^{1/2}\right]+1\right\}\left(\frac{1}{1+8\sin^2 x_m}\right)^{(m+1)/2}\\ &=8(1+8\sin^2 x_m)^{-(m+1)/2}\leqslant 1 \qquad \text{for} \qquad m\geqslant 4.\end{aligned}$$

This completes part (ii) (see (13) and (15)).

(iii) Use (6), part (i), and Theorem 2(i) and observe that

$$z_{\kappa,\nu}(1)=z_{\kappa,\nu+1}(0).$$

Remarks. (i) Since we estimated the remainder $R_\kappa(r)$ term by term with absolute values, the estimates for small κ are not too sharp and the approximation of the zeros is much better than that given by Theorem 3. Compare [3, p. 291].

(ii) The zeros of $H_{\kappa,\lambda}$ are good starting points for calculating the zeros of $f_{\kappa,\lambda}$ with Newton iteration using, e.g., formula (8) in the case of arbitrary $\kappa>0$. Compare again [3, p. 291].

References

1. W. Gawronski, On the asymptotic distribution of the zeros of Hermite, Laguerre, and Jonquière polynomials, *J. Approx. Theory* **50** (1987), 214–231.
2. W. Gawronski and U. Stadtmüller, On the zeros of power series with exponential logarithmic coefficients, *Canad. Math. Bull.* **24** (1981), 257–271.
3. W. Gawronski and U. Stadtmüller, On the zeros of Jonquière's function with a large complex parameter, *Michigan Math. J.* **31** (1984), 275–293.
4. W. B. Jurkat and A. Peyerimhoff, On power series with negative zeros, *Tôhoku Math. J.* (2) **24** (1972), 207–221.

5. D. F. LAWDEN, The function $\sum_1^\infty n^r z^n$ and associated polynomials, *Proc. Cambridge Philos. Soc.* **47** (1951), 309–314.
6. M. LERCH, Note sur la fonction $K(w, x, s) = \sum_{k=0}^\infty e^{2\pi i k x}/(w+k)^s$, *Acta Math.* **11** (1887), 19–24.
7. E. LINDELÖF, "Le calcul des résidus et ses applications à la théorie des fonctions," Gauthier–Villars, Paris, 1905.
8. W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, New York, 1966.
9. W. MIESNER AND E. WIRSING, On the zeros of $\sum (n+1)^\kappa z^n$, *J. London Math. Soc.* **40** (1965), 421–424.
10. G. MEINARDUS AND G. MERZ, Zur periodischen Spline-Interpolation, *in* "Spline Funktionen" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 177–195, Bibliographisches Institut, Mannheim, 1974.
11. H. TER MORSCHE, On the existence and convergence of interpolating periodic spline functions, *in* "Spline Funktionen" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 197–214, Bibliographisches Institut, Mannheim, 1974.
12. A. PEYERIMHOFF, On the zeros of power series, *Michigan Math. J.* **13** (1966), 193–214.
13. A. PEYERIMHOFF, "Lectures on Summability," Lecture Notes in Mathematics, Vol. 107, Springer-Verlag, Berlin, 1969.
14. G. PÓLYA AND G. SZEGÖ, "Aufgaben und Lehrsätze aus der Analysis I," Springer-Verlag, Berlin, 1970.
15. M. REIMER, Extremal spline bases, *J. Approx. Theory* **36** (1982), 91–98.
16. M. REIMER, The radius of convergence of a cardinal Lagrange spline series of odd degree, *J. Approx. Theory* **39** (1983), 289–294.
17. M. REIMER, The main-roots of the Euler–Frobenius polynomials, *J. Approx. Theory* **45** (1985), 358–362.
18. S. L. SOBOLEV, On the roots of Euler polynomials, *Dokl. Akad. Nauk SSSR* **235** (1977) [*Soviet Math. Dokl.* **18** (1977), 935–938].
19. U. STADTMÜLLER, On the zeros of power series with logarithmic coefficients, *Canad. Math. Bull.* **22** (1979), 221–233.
20. C. TRUESDELL, On a function which occurs in the theory of the structure of polymers, *Ann. of Math.* (2) **46** (1945), 144–157.
21. E. WIRSING, On the monotonicity of the zeros of two power series, *Michigan Math. J.* **13** (1966), 215–218.
22. W. WIRTINGER, Über eine besondere Dirichletsche Reihe, *J. Reine Angew. Math.* **129** (1905), 214–219.